# Design of Absolute Encoder Disk Coding Based on Affine n digit N-ary Gray Code

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Abstract—This paper presents a proposal to design the absolute encoder disk based on n digit N-ary cyclic gray code. The n digit N-ary cyclic gray code is generated using affine transformation. Based on the concept of the n digit N-ary cyclic gray code, two types of the coded tracks were designed, namely, Nary cyclic color coded track and N-ary cyclic gray scale coded track. The resolution of the proposed coded disk is compared with the traditional gray coded disk. A 5 digit 5-ary color coded disk gives approximately 100 times better resolution than the traditional binary gray coded track with same number of coded tracks and sensors.

# Keywords—Absolute encoder; n digit N-ary gray code; Affine transformation; Hamiltonian cycle; refelective sensor

#### I. INTRODUCTION

Optical Encoder is an electro-mechanical device that measures real-time angular position and converts the angle information into unique numerical code. Optical encoders are widely used in control and measurement systems, radar system, and high performance servo applications because of their high resolution, accuracy, and long lifetime. Optical Encoders are classified into incremental encoders and absolute encoders. In case of incremental encoder, the disk pattern is a uniform black and transparent printed pattern around the disk. Absolute encoder differs from the incremental encoder in the printed pattern on the disk. Absolute encoder employs multiple tracks of cyclic or unit distance codes [1].

The traditional optical absolute encoders use binary gray coded disk pattern to achieve the angular position. But these encoders suffer from the inherent problem of quantization noise on account of the limited resolution they possess [2]. The resolution of the binary gray code absolute encoder can be increased at the cost of increasing the number of tracks. With the increase of the number of tracks of the coded disk, the size and weight of the encoder also increases. With the rapid improvement in technology and the increase of the application field of absolute encoders, high precision and miniaturization of the optical encoder has become an important topic of research.

A number of studies have been performed to achieve miniaturized encoder with high resolution. To reduce the

number of coded tracks of the encoders, single track disks were proposed [3]. Some researchers have used pseudorandom coded patterns [4] or De Burjin sequence [5] to design the coded track. Also, *M*-code was used to design the absolute encoder disk pattern [6]. These methods use single coded track encoder disk to find the position. Coded track design based on the graph theory was proposed by T. Dziwinski [7]. The graph theory based track design uses two dimensional matrix decoding by the use of two dimensional image sensors.

In this paper, authors have proposed a new method to code the encoder disk to achieve high degree of resolution. The work focuses on the use of a coding approach that can change both the base and the power term of the denominator in the resolution expression of the encoder. To achieve the above mentioned goal, n digit N-ary gray code is studied. The applications of n digit N-ary gray code have been found in many different areas. Raymond et al. describes the use of Nary gray code in the field of digital circuit design [8]. N-ary gray code is used in digital communication system frequently [9]-[12]. In [13], the N-ary gray code was used for image bit plan decomposition, image denoising and encryption. Motion estimation of video processing has also used N-ary gray codes [14]. According to the characteristics of N-ary gray codes, they can be divided into different types. A survey on different types of N-ary gray codes is discussed in [15]. However all of them cannot be used to design the absolute rotary encoder disk to measure angular position. To design a rotary encoder disk, the *N*-ary code should have bijective and cyclic characteristics. In this paper, to meet these conditions, the cyclic *n* digit *N*-ary gray code based on affine transformation is considered. The use of cyclic *n* digit *N*-ary gray code to code the encoder disk opens the option to change not only the power term but also the base in the encoder resolution expression. This results in the improvement of the resolution of the absolute encoder. In addition to that, miniaturization in the overall track diameter can be achieved compared to the tradition binary encoder systems. Even though the coded disk uses same numbers of tracks and sensors as that of the traditional binary gray coded track, the resolution is higher than the traditional binary gray coded encoder.

The paper is organized into five sections: section II describes a brief overview on the existing absolute encoder



Fig. 1. Absolute encoder coded tracks with (a) binary gray code (b) gray code with vernier track.

disk coding. In section III, the proposed coding method and the mathematics behind it is discussed. Section IV presents the encoding method and the color N-ary coding of the coded disk. The detail of the decoding principle of the n digit N-ary coding is presented in V.

#### II. OVERVIEW OF THE ABSOLUTE ENCODER DISK CODING

The construction of optical encoder disk is based on cyclic code theory. There are various cyclic coding approaches to encode the coded track. Among all the different kinds of encoder disk coding approaches, disk design based on the binary gray code is the most common one. An absolute encoder coded track with 3 bit binary gray code is shown in Fig. 1(a). The encoder disk has 3 tracks with coded patterns. To decode the coded patterns, 3 photo sensors are arranged over the tracks. The positions of the photo sensors are shown by the small circles. In general, for encoders using binary gray code, to achieve n bits of resolution, n numbers of coded tracks are need. To decode n coded tracks, n numbers of sensor heads are required.

To improve the resolution of the gray code encoder, vernier principle was added to the traditional gray coded track [2]. According to the vernier principle, n divisions of the vernier scale have same length as n-1 divisions of the main scale. To apply vernier principle on the n track gray coded disk, an extra track with  $2^n$  slits are added on the outermost track. Fig. 1(b) shows 3 bit gray coded absolute encoder with the vernier track. In Table I, expressions for the resolution, number of tracks and number of sensors are mentioned for a n

TABLE I. EXPRESSIONS FOR BINARY GRAY CODE ENCODER

Codes	Gray	Gray with Vernier		
Resolution	$\frac{360^{\circ}}{2^n}$	$\frac{360^{\circ}}{2^{n-1}(2^{n-1}-1)}$		
No. of tracks	п	п		
No. of sensors	п	$(n-1) + (2^{n-1} \pm 1)$		

bit binary gray code encoder without and with vernier track, respectively. As shown in Fig. 1(b), on the outer periphery of the main gray coded track, an extra vernier track with 4 slits are added. To decode the signal from the vernier track 5 or 3 numbers of extra photo sensors are needed along with the main track sensors.

Although the binary gray code encoder posses the simple disk coding method, binary gray coded absolute encoder suffers the overall increase of the size and weight due to the increase in size of the coded disk to improve the resolution. The addition of the vernier track increases the resolution reasonably, but  $2^{n-1} + 1$  or  $2^{n-1} - 1$  numbers of sensor heads are needed to decode the vernier track along with n -1 sensor heads to decode main gray coded track as mentioned in Table I. Thus the absolute encoder coded track design using conventional binary gray code is not helpful to design a miniaturized encoder track.

#### III. PROPOSED METHOD

To tackle these problems, we have introduced the design of the coded track based on n digit N-ary gray code. Basically it is based on existing gray code. However, it is efficient and powerful way to code the encoder disk. Unlike the binary code theory in which the base term is fixed to 2, in n digit N-ary gray code, we can change the base number as we want as well as the exponent of the power representation in resolution.

The absolute encoder reads and decodes angular position with the help of the coded disk. To read different position over a complete span of  $360^\circ$ , the code pattern over the coded track needs to be distinct for a specific location. So the code must be 1) bijective and 2) cyclic code. A novel *n* digit *N*-ary gray code is analyzed to check the above mentioned conditions.

For *N*,  $n \ge 2$ , an *n*-digit *N*-ary gray code is a sequence in which each *n* digit string with digit from the set  $\{0, 1, ..., N-1\}$  occurs once [16]. Thus, *N*-ary gray code satisfies the bijective code condition. And any two consecutive strings differ in only one digit; the difference is equal to  $\pm 1$ .

**Theorem:** The distance between two strings  $x_1, x_2, \dots, x_n$  and  $y_1, y_2, \dots, y_n$  of equal length *n* over a *N*-ary alphabet  $\{0, 1, \dots, N-1\}$  of size  $N \ge 2$  is called as Lee distance. Mathematically Lee distance is defined as  $D_L$  as follows,

$$D_L(x_1....x_n, y_1....y_n) = \sum_{i=1}^n \min\{|x_i - y_i|, N - |x_i - y_i|\}$$
(1)

**Definition 1**: If the Lee distance between the last and the first string is equal to 1, the code is called cyclic code.

**Definition 2**: In graph theory, A Hamiltonian cycle in a graph G = (V, E) is a cycle that goes through every vertex exactly once.

Here V is the vertex of the graph and E is the edge between two vertices.

As mentioned above, in *N*-ary gray code consecutive strings has Lee distance equals to 1. And *N*-ary Gray code can be considered as a Hamiltonian cycle in the *N*-ary *n*-cube whose vertices are all *n*-strings with digits from set  $\{0, 1, ..., N-1\}$  in which two vertices are adjacent and differ in only one digit by  $\pm 1$ . Thus *N*-ary gray code is a cyclic code and satisfies bijection. From the above discussion, it is clear that *N*-ary gray code can be used to design a absolute rotary encoder to measure angular position.

#### IV. ENCODING PRINCIPLE

A. Encoding principle of N-ary cyclic code

Any integer i can be expressed in base N expansion as follows

$$i = i_1 N^{n-1} + i_2 N^{n-2} + \dots + i_n N^0$$
<sup>(2)</sup>

**Lemma 1**: An *n* dimensional *N*-ary gray code is a permutation *G* of the sequence vectors 0, 1,....,  $N^n$ -1 such that, if any two vectors *a* and *b* satisfies the condition

$$a - b = 1 \mod N^n \tag{3}$$

Then, 
$$G(a) - G(b) = \pm N^p$$
, for some  $0 \le p \le n-1$  (4)

To get a cyclic n dimensional N-ary gray code, affine transformation method [17-21] is considered. This method of gray code transformation doesn't depend on constrains conditions of N to be prime or even number to achieve cyclic gray code.

To make G mentioned in (4) affine, it must follow the condition mentioned below

*Lemma* 2: If  $a-b=1 \mod N^n$  then,  $a-b = \sum_{i=0}^{p} m^i$  for some  $0 \le p \le n-1$ ; thus the vector difference of a-b consists of a string of 0's followed by string of 1's.

Now, considering G to be the affine transformation  $\langle M, u \rangle$ ,

$$G(x) = x \cdot M + u \tag{5}$$

To find G, we need to know M and u. The process is mentioned below

For a and b, (5) can be rewritten as,

$$G(a) = a \cdot M + u \tag{6}$$

 $\alpha$ 

$$G(b) = b \cdot M + u \tag{7}$$

Subtracting (6) and (7),

$$G(a) - G(b) = a \cdot M + u - b \cdot M - u = \pm N^{p},$$
  
for  $0 \le p \le n - 1$ 

Thus,

$$(a-b) \cdot M = \pm N^p \tag{8}$$

Considering lemma 2 mentioned above, *n* possible cases can presented in matrix  $n \times n$  matrix form as follows,

$$\begin{bmatrix} \sum_{i=0}^{n-1} N^{i} \\ \sum_{i=0}^{n-1} N^{i} \\ \vdots \\ \vdots \\ \sum_{i=0}^{n-1} N^{i} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \vdots & \vdots \\ 0 & 1 & 1 & \vdots \\ \vdots & 0 & 1 & \vdots \\ \vdots & 0 & 1 & \vdots \\ \vdots & 0 & 0 & \vdots \\ \vdots & \vdots & 0 & 1 \end{bmatrix} = \sum_{i=0}^{n-1} Z^{i}$$
(9)

where,  $Z^i$  has 1's on first *i* th super-diagonal and 0's everywhere else.

Thus from (3), (8) and (9),

$$\left(\sum_{i}^{n-1} Z^{i}\right) M = V.Q \tag{10}$$

where, V is an  $n \times n$  matrix with 1's and -1's in as it's diagonal elements, Q is also an  $n \times n$  matrix, known as the permutation matrix.

Taking the inverse of 
$$\left(\sum_{i}^{n-1} Z^{i}\right)$$
, (10) can be rewritten as,  

$$M = \left(\sum_{i}^{n-1} Z^{i}\right)^{-1} V Q$$
(11)

$$M = (I - S). V. Q$$
 (12)

where,  $\left(\sum_{i=1}^{n-1} Z^{i}\right)^{-1} = I - S$  and *I* is the singularity matrix.

Finally, (5) can be rewritten as,

$$G(x) = ((I - S) \cdot V \cdot Q) + u \tag{13}$$

where *u* is any  $1 \times n$  *N*-ary vector; S is an  $n \times n$  matrix with 1's on its super-diagonal and 0 elsewhere; *V* is an  $n \times n$  matrix with 1's and -1's in as it's diagonal elements; *Q* is an  $n \times n$  permutation matrix. Thus using the (13), a cyclic *n* digit *N*-ary gray code can be obtained. For a pair of (n, N), N > 2,

Decimal	Triple	Ternary cyclic gray code	Color code
0	000	000	RRR
1	001	001	RRG
2	002	002	RRB
3	010	012	RGB
•	•	•	•
•	•	•	•
23	212	221	BBG
24	220	201	BRG
25	221	202	BRB
26	222	200	BRR



Fig. 2. Color 3-digit 3-ary cyclic coded track.

TABLE III. ANGLE REPRESENTATION OF THE CODED PATTERN FOR 3-DIGIT 3-ARY GRAY CODE

Position	Code	Angle(°)		
0	RRR(000)	0		
1	RRG(001)	13.33		
2	RRB(002)	26.66		
3	RGB(012)	39.99		
	•			
•	•	•		
•	•	•		
26	BRR	346.58		

 $2^n \cdot n! \cdot N^n$  numbers of affine gray code transformations is possible.

# B. Example

Let us consider the formation of 2-digit 3-ary gray code using the principle mentioned above. With n = 2, N=3, the possible numbers of affine gray code transformations is,

$$2^{n} \cdot n! \cdot N^{n} = 2^{2} \cdot 2! \cdot 3^{2} = 72$$

For 
$$n = 2$$
,  $N=3$ , taking  $V = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ ,  $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$  and  $v = 6$ ;

# **Ternary representation**

3-ary gray

0		00				$\left\lceil 20 \right\rceil$		20
1		01				20		21
2		02				20		22
3		10	Г1	<b>2</b> ∃ [2	07	20		11
4	≈	11			$\frac{1}{2}$ +	20	=	10
5		12		I	2	20		11
6		20				20		02
7		21				20		01
8		22				20		00

The ternary representation is equivalent to the BCD code in case of a binary system of notation. From the above example, the 2 digit 3-ary gray code satisfies the bijection and is cyclic in nature. Thus n digit N-ary gray can be used to design coded track for absolute encoder.

### C. Color N-ary cyclic code

*n* digit *N*-ary gray code can be represented as a combination of different colors. For an example, we can represent a 3 digit 3-ary gray code as a combination of red, green and blue colors. Red, green and blue can be considered as 0, 1, and 2, respectively. Table II shows the representation of the ternary code in color code. As shown in the Table II, a ternary cyclic gray code can divide  $360^{\circ}$  into 27 distinct divisions.

Thus using an *n*-digit *N*-ary cyclic gray code, the resolution can be expressed as,

Resolution = 
$$\frac{360^{\circ}}{N^n}$$
 (14)

A ternary 3 digit cyclic color code will give a resolution of 13.33 °. Fig. 2 shows a ternary color cyclic coded track. And with the increase of the N value, the resolution can be improved. Table III represents the relation between the coded track and the angular positions for a 3 digit 3 ary gray coded track. In Table IV a comparison of resolution between traditional gray code track and the *n*-digit *N*-ary gray code is made. Even though the number of tracks is equal to the traditional gray code, superior resolution can be achieved if Nary gray code is used. For an example, a color 5-ary 5-digit gray code gives approximately 100 times better resolution compared to the traditional binary gray coded track. To achieve approximately same amount of resolution using a traditional binary gray coded track, 6 extra binary gray coded tracks are needed. Thus, using N-ary gray code, number of coded tracks can be reduced by 6. So it results in reduction of overall coded disk diameter. In general, under the condition of equal number of digits, the *n* digit *N*-ary gray coded disk gives  $(N/2)^n$  times better resolution compared with the traditional gray coded disk. To read the angular positions over a full rotation of the coded disk, color recognition sensors can be used.

N N	1	2	3	4	5	 п
2	180°	90°	45°	22.5°	11.25°	 $\frac{360^{\circ}}{2^{n}}$
3	120°	40°	13.33°	4.44°	1.48°	 $\frac{360^{\circ}}{3^n}$
4	90°	22.50°	5.62°	1.40°	0.35°	 $\frac{360^{\circ}}{4^n}$
5	72°	14.40°	2.88°	0.57°	0.11	 $\frac{360^{\circ}}{5^{n}}$
•	•	•	•	•	•	
•	•	•	•	•	•	 •
•	•	•	•	•	•	 •
N	$360^{\circ}/_{N}$	$\frac{360^{\circ}}{N^{2}}$	$\frac{360^{\circ}}{N^{3}}$	$\frac{360^{\circ}}{N^{4}}$	$\frac{360^{\circ}}{N^{5}}$	 $\frac{360^{\circ}}{N^n}$

 TABLE IV.
 COMPARISON OF THE RESOLUTION AMONG *n* -DIGIT *N*-ARY

 GRAY CODED TRACKS
 GRAY CODED TRACKS



Fig. 3. Gray scale representation of the color 3-digit 3-ary track.



Fig. 4. (a) Decoding method of the n digit N ary gray code (b) Conversion method of addition result of the decoder.

# D. Color n digit N-ary cyclic code in gray scale.

Fig. 3 shows the gray scale track design of the color track shown in Fig. 2. The gray scale track can be obtained by single color channel method, using the data from a single color channel. It is the fastest computation method for gray scale conversion. For example, in Fig. 3, red color channel is used to transform the colored track. The red segment in the color track is transformed to the brightest segment in the gray scale track. The other two segments, blue and green are also transformed to different gray levels depending on the brightness in them. The gray scale track can be decoded using reflecting sensors like CNY-70. In gray scale track with variation of the gray level from brightest point to the darkest point, results in the variation of the amount of reflected light. This can be read by the reflective sensors. Instead of designing a color track, use of a gray scale track results in reduction in the cost of the color sensors.

#### V. DECODING PRINCIPLE

The decoding of the *n* digit *N*-ary gray code can easily be done using simple electronic circuit. Fig. 4(a) shows the decoding method of a 3 digit 3-ary gray code. The decoder consists of adder and the bits are introduced starting with MSB to LSB. As shown in Table II, we have 000 and 222 bit patterns for a 3 digit 3-ary gray code. So, using the adder circuit, the minimum and the maximum achievable addition values are 0 and 6. But we have 3 digits 0, 1, 2 available. So during the decoding process, if the adder circuit gives addition result greater than 2, they were substituted according to the fashion shown in Fig. 4(b). The decoding process is illustrated in the following examples,

#### Example:

Case 1: Let us consider the given 3 digit 3 ary gray code is 011.

Here, 
$$g_3 = 0$$
,  $g_2 = 1$ ,  $g_1 = 1$ 

So,  $A_3 = 0$ ;  $A_2 = g_2 + g_3 = 1$ ;  $A_1 = (g_2 + g_3) + g_1 = 2$ 

Finally the triple is 012; this is equivalent to decimal 5

*Case 2*: Let us consider the given 3 digit 3 ary gray code is 212.

Here, 
$$g_3 = 2$$
,  $g_2 = 1$ ,  $g_1 = 2$ 

So, 
$$A_3 = 2$$
;  $A_2 = g_2 + g_3 = 3$ ;  $A_1 = (g_2 + g_3) + g_1 = 5$ 

From Fig. 4, we can write,

 $A_3 = 2; A_2 = 0; A_1 = 2$ 

Finally the triple is 202; this is equivalent to decimal 20. Triples here are equivalent to the BCD code in a binary system. Thus the n digit N-ary gray code can easily be decoded and can be implemented in real system.

## VI. CONCLUSION

A new type of disk design of the rotary absolute encoder based on *N*-ary cyclic gray code is proposed in this paper. The proposed coded disk design results in miniaturization of the coded track with improved resolution compared to the conventional gray code tracks. The proposed encoder coded track design has enabled the use of base as well as power term of the denominator in the encoder resolution expression. Under the condition of equal number of n digit, the proposed *N*-ary gray coded disk gives  $(N/2)^n$  times better resolution compared with the traditional gray coded disk. In addition, the coded disk can easily be decoded using cheap sensing system and electronic circuit. In further research, the design of the single track *N*-ary gray code will be considered, along with that, the sensing system of the coded disk and the data acquisition system will be designed.

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